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Extension of Otolorin's Scheme to a System Of Nonlinear Equations with Singular Jacobian

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Abstract:- In this paper, we extend the idea of Otolorin's [1] for solving a single variable nonlinear equation to a system of non linear equations, Permitting Singular Jacobian at some points in the vicinity of the required root. It is proved that this modification has quadratic convergence. Using this modification, we further presented new cubically convergent Predictor-corrector type method free from second order derivative. The proposed algorithms are simple and straight forward to implement, Several numerical examples are given to illustrate their efficiency and the performance of the presented methods Therefore, these methods may be viewed as an extension and generalization of the existing methods.

Keywords:- Nonlinear equations, Iterative's methods, Taylor series expansion, Singular Jacobian, Predictor-corrector method Order of Convergence.

1.Introduction

Due to the fact that system of nonlinear equations arises frequently in science and engineering they have attracted researcher's interest. For example, nonlinear system of equations, after the necessary processing step of implicit discretization, is solved by finding the solutions of system of equations. We consider here the problem of finding a real zero, say $x^* = (x^*_1, x^*_2 \ldots\ldots; x^*_n)^T$, of a system of non linear equations

$$f_1(x_1, x_2, \ldots\ldots, x_n) = 0;$$
$$f_2(x_1, x_2, \ldots\ldots, x_n) = 0;$$
$$\ldots\ldots;$$
$$\ldots\ldots;$$
$$f_n(x_1, x_2, \ldots\ldots, x_n) = 0;$$

This system can referred in vector form by

$$F(X) = 0 \quad (1.1)$$

where $F = (f_1, f_2, \ldots\ldots, f_n)^T$ and $X = (x_1, x_2, \ldots\ldots, x_n)^T$

Let the mapping $F : D \subseteq R^n \to R^n$ assumed to satisfy the following assumptions (1.1) F(X) is continuously differentiable in an open neighborhood D of X*. There exists a solution vector X*of (1.1) in D such that $F(X^*) = 0$ and $F(X^*) \neq 0$ then the standard method for finding the solution to equations (1.1) is the classical Newton method [2-5] given by

$$X^{(k+1)} = X^k - F(X^k)/F'(X^k); \quad k = 0, 1, \ldots\ldots (1.2)$$

Though, Newton formula (1.2) is simple and fast with quadratic convergence but it may fail miserably if at any stage of computation, the Jacobian matrix of F(X) at any iterative point is singular or almost singular i.e. $|F'(X)| = 0$ Therefore, it has poor convergence and stability problems as it is very sensitive to initial guess. Classical methods by the simple modification of iteration processes.

2. Definition Of Some Means

For given real numbers a and b, some well-known means (only free from square root) is defined as numbers :

A(Arithmetic Mean) = $(a + b)/2$

C(Centre - Harmonic Mean) = $(a^2 + b^2)/(a + b$

H(Centroidal Mean) = $2(a^2 + ab + b^2)/3(a + b)$

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3. Development of the Method

First we shall outline the derivation of the method in n-dimensions. If we multiply by $e^{p(x_1 - x_{10})}$, $e^{p(x_2 - x_{20})}$, ……….. $e^{p(x_n - x_{n0})}$; each of above expression the result will not be change. Consider the following system of n-nonlinear equations in n-unknowns $x_1, x_2, \ldots\ldots, xn$:

$$e^{p(x_1 - x_{10})} f_1(x_1, x_2, \ldots\ldots, x_n) = 0;$$
$$e^{p(x_2 - x_{20})} f_2(x_1, x_2, \ldots\ldots, x_n) = 0;$$
$$\ldots\ldots;$$
$$\ldots\ldots;$$
$$e^{p(x_n - x_{20})} fn(x_1, x_2, \ldots\ldots, x_n) = 0;$$

Taking an estimate($x_{10}, x_{20}, \ldots, x_{n0}$) of a solution (1.1), we try to compute step size($h_1, h_2, \ldots, h_n$) such that

$$X = (x_{10} + h_1, x_{20} + h_2, \ldots, ; x_{n0} + h_n)^T \quad (3.1)$$

Let us denote the approximation as $X = (x_1, x_2, \ldots\ldots, x_n)^T$ and the step size as $H = (h_1, h_2, \ldots\ldots, h_n)^T$

Using Taylor's therom for n-variables in equation (1.1), we get $F(X) = F(X_0 + H) = F(X_0) + HF'(X_0) + (1/2!)H^2F''(X_0) + \ldots\ldots$ (3.2)

$X = X_0 + H$ where $X_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$

$0 = f_1(X)$

$= f1(X0 + H)$

$\approx [f1(X0) + h_1\partial f_1/\partial x_1 + h_2\partial f_1/\partial x_2 + \ldots\ldots + h_n\partial f_1/\partial x_n + O(h^2)]e^{ph_1}$

$= [f_1(X_0) + h_1\partial f_1/\partial x_1 + h_2\partial f_1/\partial x_2 + \ldots\ldots + hn\partial f_1/\partial x_n + O(h^2)](1 + ph_1 + O(h^2)$

$= f_1(X_0) + h_1\partial f_1/\partial x_1 + h_2\partial f_1/\partial x_2 + \ldots\ldots + h_n\partial f1/\partial xn + ph_1f_1(X_0) + O(h^2)$

Similarly

$f_2(X_0) + h_1\partial f_2/\partial x_1 + h_2\partial f_2/\partial x_2 + \ldots\ldots + h_n\partial f_2/\partial x_n + ph_2f_2(X_0) + O(h^2) = 0$

$f_3(X_0) + h_1\partial f_3/\partial x_1 + h_2\partial f_3/\partial x_2 + \ldots\ldots + h_n\partial f_3/\partial x_n + ph_3f_3(X_0) + O(h^2) = 0$

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$f_n(X_0) + h_1\partial f_n/\partial x_1 + h_2\partial f_n/\partial x_2 + \ldots\ldots + h_n\partial f_n/\partial x_n + ph_n f_n(X_0) + O(h^2) = 0$

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Converting the system in matrix form, we have

$$\begin{bmatrix} \frac{\partial f1}{\partial x1} & \cdots\cdots\cdots & \frac{\partial f1}{\partial xn} \\ \cdots\cdots.. & \cdots\cdots.. & \cdots\cdots. \\ \frac{\partial fn}{\partial x1} & \cdots\cdots\cdots & \frac{\partial fn}{\partial xn}\end{bmatrix}\begin{bmatrix}h1\\h2\\..\\..\\hn\end{bmatrix} +$$

$$\begin{bmatrix} pf1(X0) & 0 & \cdots & 0 \\ \cdots\cdots\cdots & & \cdots. & .. \\ 0 & 0 & \cdots & .pfn(X0).\end{bmatrix}\begin{bmatrix}h1\\h2\\..\\..\\hn\end{bmatrix} + \begin{bmatrix}f1(X0)\\f2(X0)\\..\\..\\fn(Xn)\end{bmatrix}=0$$

$$[J_{F(X_0)} + \mathrm{diag}\,(pf_i(X_0))]\begin{bmatrix}h1\\h2\\..\\..\\hn\end{bmatrix} = -\begin{bmatrix}f1(X0)\\f2(X0)\\..\\..\\fn(Xn)\end{bmatrix}$$

$$\begin{bmatrix}h1\\h2\\..\\..\\hn\end{bmatrix} = -\frac{F(X_0)}{[J_{F(X_0)} + \mathrm{diag}(pf_i(X_0))]}$$

As $X = X_0 + H$, we have

$X = X_0 - [J_{F(X_0)} + \mathrm{diag}\,(pfi(X_0))]^{-1}F(X_0)$ (3.3)

Where p is any real non-zero parameter.

If p = 0 then given method is Newton's method. Therefore p is a non-zero real number chosen so that the denominator has maximum value.

$$P=\begin{cases} -m^2 if\ |J_{F(X0)}| < 0\ and\ |diag(pf_i(X0)| > 0 \\ -m^2 if\ |J_{F(X0)}| > 0\ and\ |diag(pf_i(X0)| < 0 \\ m^2 if\ |J_{F(X0)}| < 0\ and\ |diag(pf_i(X0)| < 0 \\ m^2 if\ |J_{F(X0)}| > 0\ and\ |diag(pf_i(X0)| > 0 \end{cases}$$

4. Families of predictor -corrector type methods

Using formula (3.3), we shall propose various families of modified predictor type methods. The following lemma will be used to describe the modified variants of Newton method; its proof can be found in [3].

Lemma4.1. Let $F: D \subseteq R^n$ be continuously differentiable on an open interval D then for

Any X, Y ϵ D F(X) satisfies

$F(Y) - F(X) = \int_0^1 F'(x + t(Y - X))(Y - X)\,dt$ (4.1)

Once an iterate(x0)is obtained ,using (4.1) we have

$F(Y) = F(X) + \int_0^1 F'(x + t(Y - X))(Y - X)\,dt$ (4.2)

If an estimate of integral (4.2) is made by means of the midpoint rule and $Y = X^@$ is taken then

$0 \approx F_0(X_0) + F_0((X_0 + X^@)/2)\,(X^@ - X_0)$ (4.3)

Is obtain and a new approximation X of $X^@$ is given by

$X = X0 - [F_0((X_0 + X^@)/2)]^{-1}F(X_0)$ (4.4)

In order to avoid implicit problem we use $(k+1)^{th}$ iteration of modification Newton method (1.2) in the right hand side of equation(4.4).Therefore the general formula for modification midpoint Newton method will be given by

$X = X_0 - [F_0((X_0+Z/2)]^{-1}F(X_0);$ (4.5)

Where $Z = X_0 - [J_{f(X0)} + dig(pf_i(X_0))]^{-1}F(X_0)$

(4.4) can be written as

$X = X0 - [J_{F((X0+Z)/2)}]^{-1}F(X0)$ (4.6)

If an estimation of integral (4.2) is made by the trapezoidal rule, then we obtain the following modified Newton formula:

$X = X_0 - \frac{2F(X0)}{F'(X0)+F'(Z)}$ (4.7)

Where $Z = X_0 - [J_{F(X0)} + dig\ (pf_i(X_0))]^{-1}F(X_0)$

Note that for p = 0, formula is the arithmetic mean Newton formula given by

$X = X_0 - \frac{2F(X0)}{F'(X0)+F'(Z)}$ (4.8)

Where $Z = X_0 - \frac{F(X0)}{F'(X0)}$

Formula (4.7) used the arithmetic mean of $F_0(X_0)$ and $F_0(Z)$ instead of $F_0(X_0)$.Therefore, this formula may be called modified arithmetic mean Newton formula Similarly by using different approximations to the indefinite integral (4.3), different iterative formula can be obtained for solving system of nonlinear equations.

Some other formulae based on means

Some other new third-order methods based on contra-harmonic mean and centroidal mean can also be obtained from formula (4.7) as follows:

(i) Modified contra-∂harmonic mean family of Newton's method

$X = X_0 - \frac{F'(X0)+F'(Z)}{F^{2'}(X0)+F^{2'}(Z)}F(X_0)$ (4.9)

Where $Z = X_0 - [J_{F(X0)} + dig(pfi(X0))]^{-1}F(X_0)$

(ii) Modified centroidal mean family Newton's method

$X = X_0 - \frac{3[F'(X0)+F'(Z)]}{2[F^{2'}(X0)+F^{2'}(Z)+F'(X0)F'(Z)]}F(X_0)$ (4.10)

Where $Z = X_0 - [J_{F(X0)} + dig(pfi(X_0))]^{-1}F(X_0)$

These modified variants of Newton method will work even if |F0(X)| is small or zero in the vicinity of the required root. Note that for p = 0, formula (4.9) and (4.10) reduce to contra-harmonic and centroid mean Newton formulas.

5. Convergence analysis

We shall present the mathematical proof for the order of convergence of formula (3.3) and (4.7) and the order of convergence for the remaining algorithms can be proved in same lines.

Theorem5.1. Let r ЄD be a simple zero of a sufficiently differentiable function $F : D \subseteq R^n \rightarrow R^n$ for interval D. Let X0 be an initial guess sufficiently closed to r and $[J_{F(X0)} + diag(pf_i(X_0))] \neq 0$ in D. Then the sequence generated by formula (3.3) is quadratical convergent.

Proof. : Let r be a simple zero of F(X) = 0 (i.e. F(r) = 0) and F : $R^n \rightarrow R^n$ be defined by $ø(X) = X - [J_{F(X)} + diag(pf_i(X))]^{-1}F(X)$

This can be written as

$[J_{F(X)} + diag(pf_i(X))] (ø(X) - X) + F(X) = 0$ (5.1)

Taking $\sum$

We have

$\sum_1^n [J_{F(X)} + diag(pf_i(X))] (ø_i(X) - X) + f_i(X) = 0$ (5.2)

Putting X = r in (5.2) we have

$ø_j(r) = r_j$ (5.3)

Differentiate (5.2) w.r.t. x_q we have

$$\sum_1^n \frac{\partial[J_{F(X)} + diag(pf_i(X))]}{\partial xq}(ø_i(X) - x_j) + \sum_1^n [J_{F(X)} + diag(pf_i(X))] \frac{\partial ø_i(X)}{\partial xq} - \delta_{jq} + \partial f_i(X)/\partial x_q = 0 \quad (5.4)$$

Using (5.3) in (5.4) and putting X = r we have

$\partial ø_j(r)/\partial xq = 0$ (5.5)

Differentiate (5.4) w.r.t. x_r we have

$$\sum_1^n \frac{\partial^2[J_{F(X)} + diag(pf_i(X))]}{\partial x_r \, \partial x_q}(ø_i(X) - x_j)$$
$$+ \sum_1^n \partial[J_{F(X)} + diag(pf_i(X))] / \partial x_q \frac{\partial ø_i(X)}{\partial xr} - \delta_{jr}$$
$$+ \sum_1^n \partial[J_{F(X)} + diag(pf_i(X))] / \partial x_r \frac{\partial ø_i(X)}{\partial xq} - \delta_{jq}$$
$$+ \sum_1^n [J_{F(X)} + diag(pf_i(X))] \frac{\partial^2 ø_i(X)}{\partial x_r \, \partial xq}$$
$$+ \partial^2 f_i(X)/\partial x_r \partial x_q = 0 \quad (5.6)$$

Putting X = r in (5.6) and using (5.5) and (5.3) we have

$\partial^2 ø_i(r)/\partial x_r \partial x_q \neq 0$ (5.7)

therefore the equation (3.3) and (4.7) is quadraticaly convergent.

Theorem5.2. Under the hypothesis of theorem (5.1) system (4.9) and (4.10) convergent to r with convergence order three.

Proof: Let us consider r ϵ R^n of F(X) = 0 as a simple zero .Let ø: $R^n \rightarrow R^n$ defined as

$ø(X) = X - inv[J_{F((X+Z)/2)}]F(X)$

or

$J_{F((X+Z)/2)}(ø(X) - X) + F(X) = 0$ (5.8)

Taking $\sum$

$\sum_1^n J_{F((X+Z)/2)} (ø_j(X) - x_j) + f_i(X) = 0$ (5.9)

Put X = r in (5.9) we have

$ø_j(r) = r_j$ (5.10)

Differentiate (5.9) w.r.t. x_q we have

$$\sum_1^n \partial J_{F((X+Z)/2)} / \partial x_q (ø_j(X) - x_j)$$
$$+ \sum_1^n J_{F((X+Z)/2)} \partial ø_j(X)/\partial x_q - \delta_{jq}) + \partial f_i(X)/\partial x_q = 0 \quad (5.11)$$

Put X = r in (5.11) and using (5.10) we have

$\partial ø_j(r)/\partial x_q = 0$ (5.12)

Differentiate (5.11) w.r.t. x_r we have

$$\sum_1^n \partial^2 J_{F((X+Z)/2)} / \partial x_r \partial x_q (ø_j(X) - x_j)$$
$$+ \sum_1^n \partial J_{F((X+Z)/2)} / \partial x_q (\partial ø_j(X)/\partial x_r - \delta_{jr})$$
$$+ \sum_1^n \partial J_{F((X+Z)/2)} / \partial x_r \, \partial ø_j(X)/\partial x_q - \delta_{jq})$$
$$+ \sum_1^n J_{F((X+Z)/2)} \partial^2 ø_j(X)/\partial x_r \partial x_q$$
$$+ \partial^2 f_i(X)/\partial x_r \partial x_q = 0 \quad (5.13)$$

Put X = r in (5.13) and using (5.10) and (5.12) we have

$\partial^2 ø_j(X)/\partial x_r \partial x_q = 0$ (5.14)

Differentiate (5.13) w.r.t. x_l we have

$$\sum_1^n \partial^3 J_{F((X+Z)/2)} / \partial x_l \partial x_r \partial x_q (ø_j(X) - x_j)$$
$$+ \sum_1^n \partial^2 J_{F((X+Z)/2)} / \partial x_r \partial x_q \, \partial ø_j(X)/\partial x_l - \delta_{jl}$$
$$+ \sum_1^n \partial^2 J_{F((X+Z)/2)} / \partial x_l \partial x_q (\partial ø_j(X)/\partial x_r - \delta_{jr})$$
$$+ \sum_1^n \partial J_{F((X+Z)/2)} / \partial x_q \, \partial^2 ø_j(X)/\partial x_l \partial x_r$$
$$+ \sum_1^n \partial^2 J_{F((X+Z)/2)} / \partial x_l \partial x_r \, \partial ø_j(X)/\partial x_q - \delta_{jq})$$
$$+ \sum_1^n \partial J_{F((X+Z)/2)} / \partial x_r \, \partial^2 ø_j(X)/\partial x_l \partial x_q$$
$$+ \sum_1^n \partial J_{F((X+Z)/2)} / \partial x_l \, \partial^2 ø_j(X)/\partial x_r \partial x_q$$
$$+ \sum_1^n J_{F((X+Z)/2)} \partial^3 ø_j(X)/\partial x_l \partial x_r \partial x_q$$
$$+ \partial^2 f_i(X)/\partial x_r \partial x_q = 0 \quad (5.15)$$

Put X = r in (5.15) and using (5.10), (5.12) and (5.14) we have

$\partial^3 ø_j(X)/\partial x_l \partial x_r \partial x_q \neq 0$ (5.16)

6. Numerical results

In this section, we shall check the performance of the Present families say S1(3.3), SS1(4.7), SB1(4.9), SSB1(4.10) and S2(3.3), SS2(4.7), SB2(4.9), SB2(4.10) by taking m = 0.5 and m = 0.05.The comparisons is carried out With Newton method and with GTM1 and GTM2[8].A mat lab program has been written to implement these methods . We use the following stopping criteria for computer programs. We use $\epsilon = e^{-10}$

(i) $|F(Xn)| < \epsilon$

For every method, we analyze the number of iterations needed to converge to the required solution. The numerical results are reported in the table 1

We consider the following problems for a system of nonlinear equations.

Problem (a)

$$x_1^2 - 2x_1 - x_2 + 0.5 = 0$$
$$x_1^2 + 4x_2^2 - 4 = 0$$

Problem (b)

$$x_1^2 + x_2^2 - 1 = 0$$
$$x_1^2 - x_2^2 + 0.5 = 0$$

Problem (c

$$x_1^2 - x_2^2 + 3\log(x_1) = 0$$
$$2x_1^2 - x_1 x_2 - 5x_1 + 1 = 0$$

Problem (d)

$$e^{x_1} + x_1 x_2 - x_2 - 0.5 = 0$$
$$\sin(x_1 x_2) + x_1 + x_2 - 1 = 0$$

Problem (e)

$$x_1 + 2x_2 - 3 = 0$$
$$2x_1^2 + x_2^2 - 5 = 0$$

Problem (f)

$$x_1^2 - x_2^2 - 4 = 0$$

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$$x_1^2 + x_2^2 - 16 = 0$$

Problem (g)

$$x_1^2 - x_2^2 - 1 = 0$$
$$x_1^3 x_2^2 - 1 = 0$$

Solution (a)
$r = (1.9006767263670658, 0.31121856541929427)^T$
Solution (b)
$r = (0.500000000000000000, 0.8660254378443865)^T$
$r = (-0.5000000000000000, -0.8660254378443865)^T$
Solution (c)
$r = (1.3192058033298924, -1.6035565551874148)^T$
Solution (d) $r = (0, 1)^T$
Solution (e)
$r = (1.4880338717125849, 0.75598306414370757)^T$
Solution (f)
$r = (3.1622776601683820, 2.44948974278312840)^T$
Solution (g)
$r = (1.2365057033915010, 0.7272869822289620)^T$

7. Conclusions

The presented families are simple to understand, easy to program and have the same rate of convergence as Newton method and GTM1, GTM2 have. The behavior of existing iterative scheme and proposed modification can be compared by their corresponding correction factors. The correction factor $F(X^k)/F'(X^k)$, which appears in the Newton method and its variants, is now modified by

$$\frac{F(X^k)}{j_{F(X^k)} + diag(pf_i(X^k))}; \quad p \neq 0$$

This factor is always well defined, even if Jocabian determinate of $|F(X^k)| = 0$ at some points in the vicinity of required root. Moreover, they have the same efficiency indices as that of existing method. Therefore, these techniques have a definite practical utility. However, if $|F'(X^k)| = 0$ and any of $f_i(X^k)$ is zero at initial guess, then the method will be work.

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Table 1: Numerical results of problems (a) to (g) using different methods.

F(X)	X	NM	GMT1	GMT2	S1	SS1	SB1	SSB1	S2	SS2	SB2	SSB2
(a)	$(3,2)^T$	9	5	6	4	2	4	3	5	3	4	3
(a)	$(1.6,0)^T$	8	6	5	3	2	3	3	4	2	3	3
(b)	$(.7,-2)^T$	7	5	5	3	2	2	2	2	3	2	2
(b)	$(-1,-2)^T$	8	6	6	5	3	3	3	4	2	3	3
(c)	$(.91,-2)^T$	8	6	5	4	2	4	2	4	2	4	2
(c)	$(1.5,-1.5)^T$	7	5	5	3	2	3	2	3	2	3	2
(d)	$(.9, .9)^T$	8	5	5	4	2	4	3	4	2	4	3
(d)	$(-0.1, 0.2)^T$	7	5	5	3	2	3	2	3	2	3	2
(e)	$(.9, .5)^T$	8	6	5	4	2	3	2	4	2	3	2
(e)	$(1.5, 1)^T$	7	5	4	3	1	2	2	3	1	2	2
(f)	$(0, 0)^T$	Fails	Fails	Fails	4	3	4	4	Div	7	9	10
(f)	$(3, 2)^T$	3	Fails	2	3	1	2	2	3	2	2	2
(g)	$(0, 0)^T$	Fails	Fails	Fails	15	13	Div	Div	26	Div	Div	Div
(g)	$(1, 2)^T$	12	12	12	12	11	12	12	12	11	12	12